Scientific Computing: The Fast Fourier Transform

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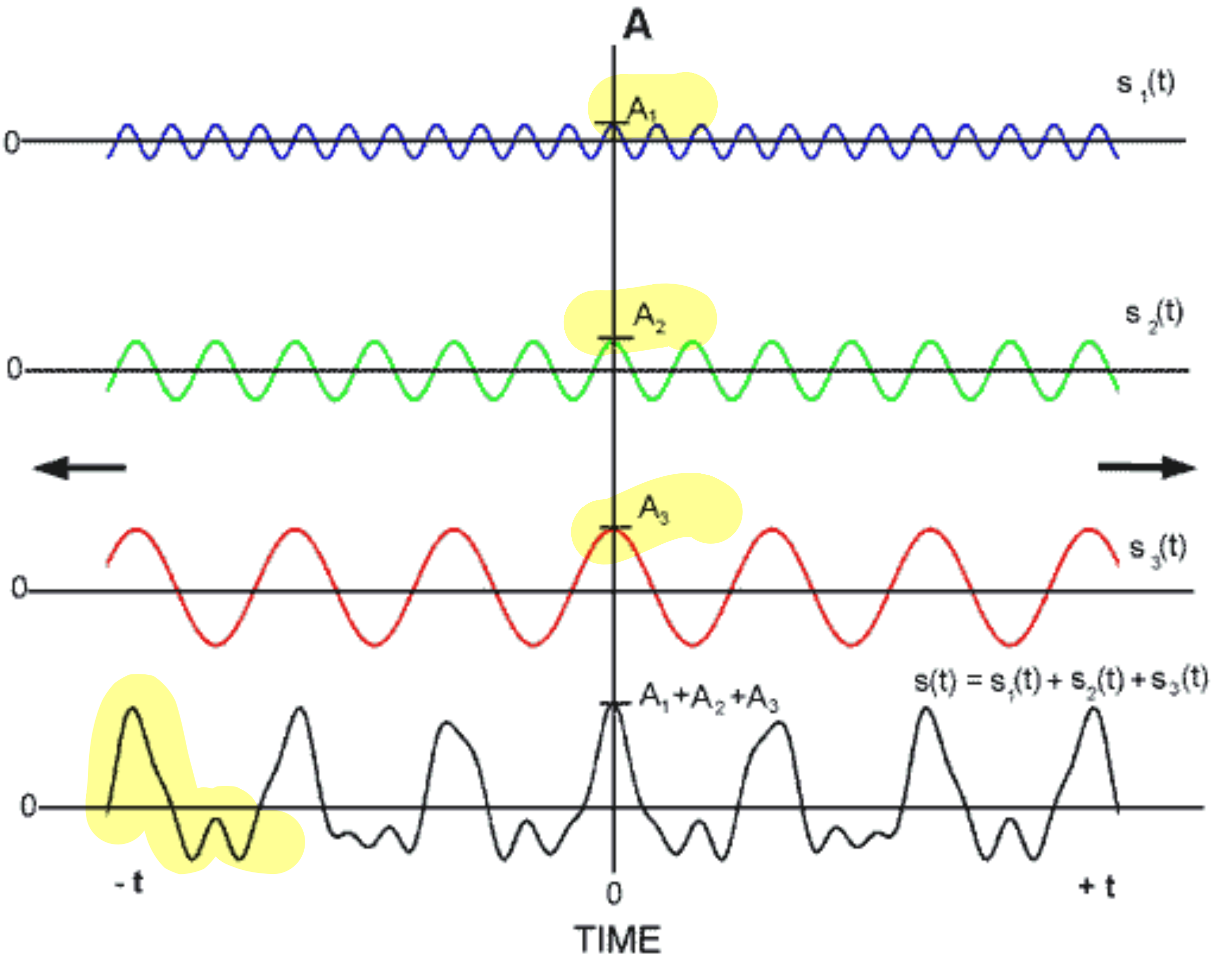
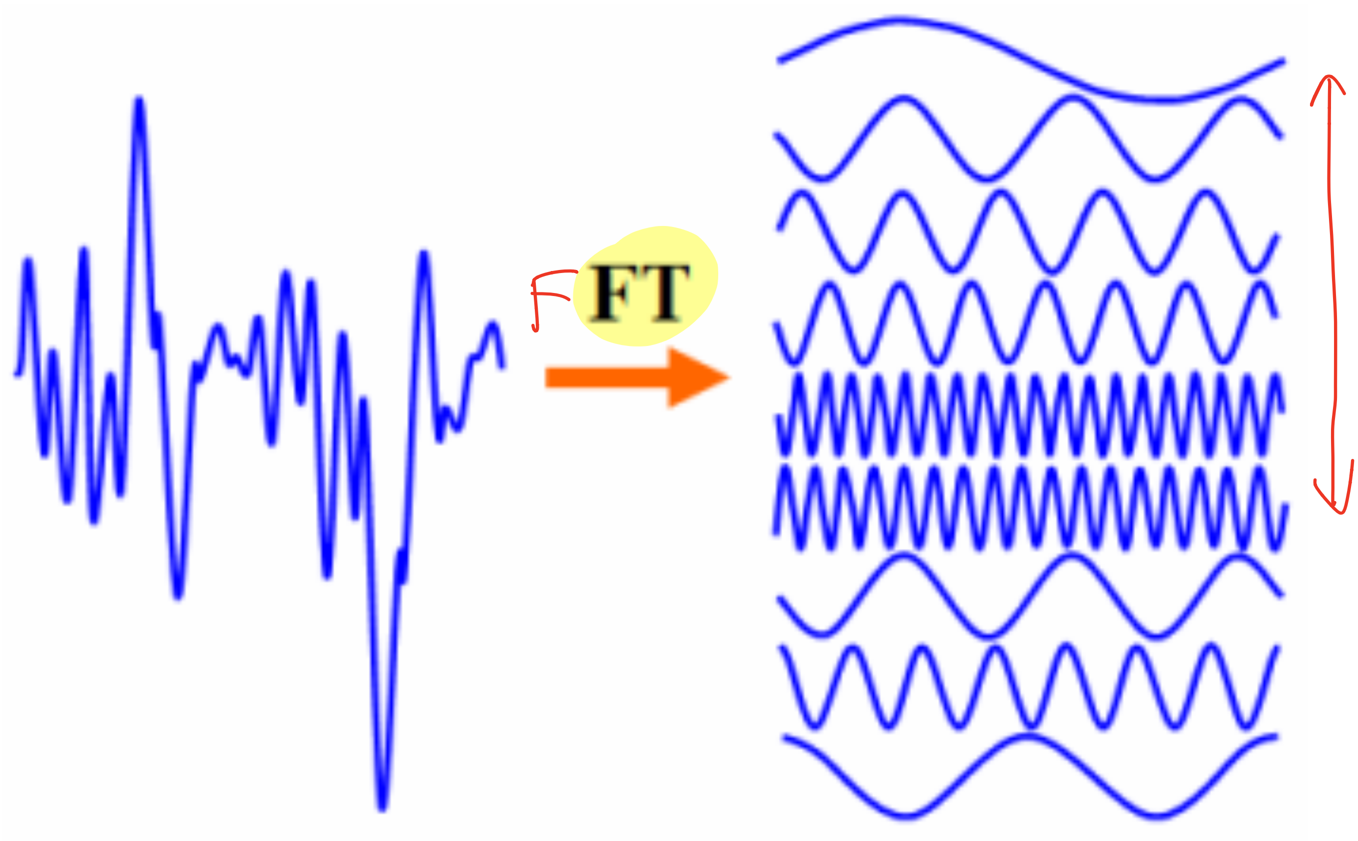
Outline

1. Fourier Series
2. Discrete Fourier Transform
3. Fast Fourier Transform
4. Applications of FFT
5. Wavelets
6. Conclusions

Fourier Composition



Fourier Decomposition



Periodic Functions

- Consider now interpolating / approximating **periodic functions** defined on the interval $I = [0, 2\pi]$:
$$\forall x \quad f(x + 2\pi) = f(x),$$
as appear in practice when analyzing signals (e.g., sound/image processing).
- Also consider only the space of complex-valued **square-integrable functions** $L^2_{2\pi}$,
$$\forall f \in L^2_w : \quad (f, f) = \|f\|^2 = \int_0^{2\pi} |f(x)|^2 \, dx < \infty.$$
- Polynomial functions are not periodic and thus basis sets based on orthogonal polynomials are not appropriate.
- Instead, consider sines and cosines as a basis function, combined together into **complex exponential functions**
$$\phi_k(x) = e^{ikx} = \cos(kx) + i\sin(kx), \quad k = 0, \pm 1, \pm 2, \ldots$$

Fourier Basis Functions

$$\phi_k(x) = e^{ikx}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- It is easy to see that these are **orhogonal** with respect to the continuous dot product

complex conjugate

$$(\phi_j, \phi_k) = \int_{x=0}^{2\pi} \phi_j(x)\phi_k^\star(x)dx = \int_0^{2\pi} \exp\left[i(j-k)x\right]dx = 2\pi\delta_{jk}$$

- The complex exponentials can be shown to form a complete **trigonometric polynomial basis** for the space $L^2_{2\pi}$, i.e.,

L_2 sense

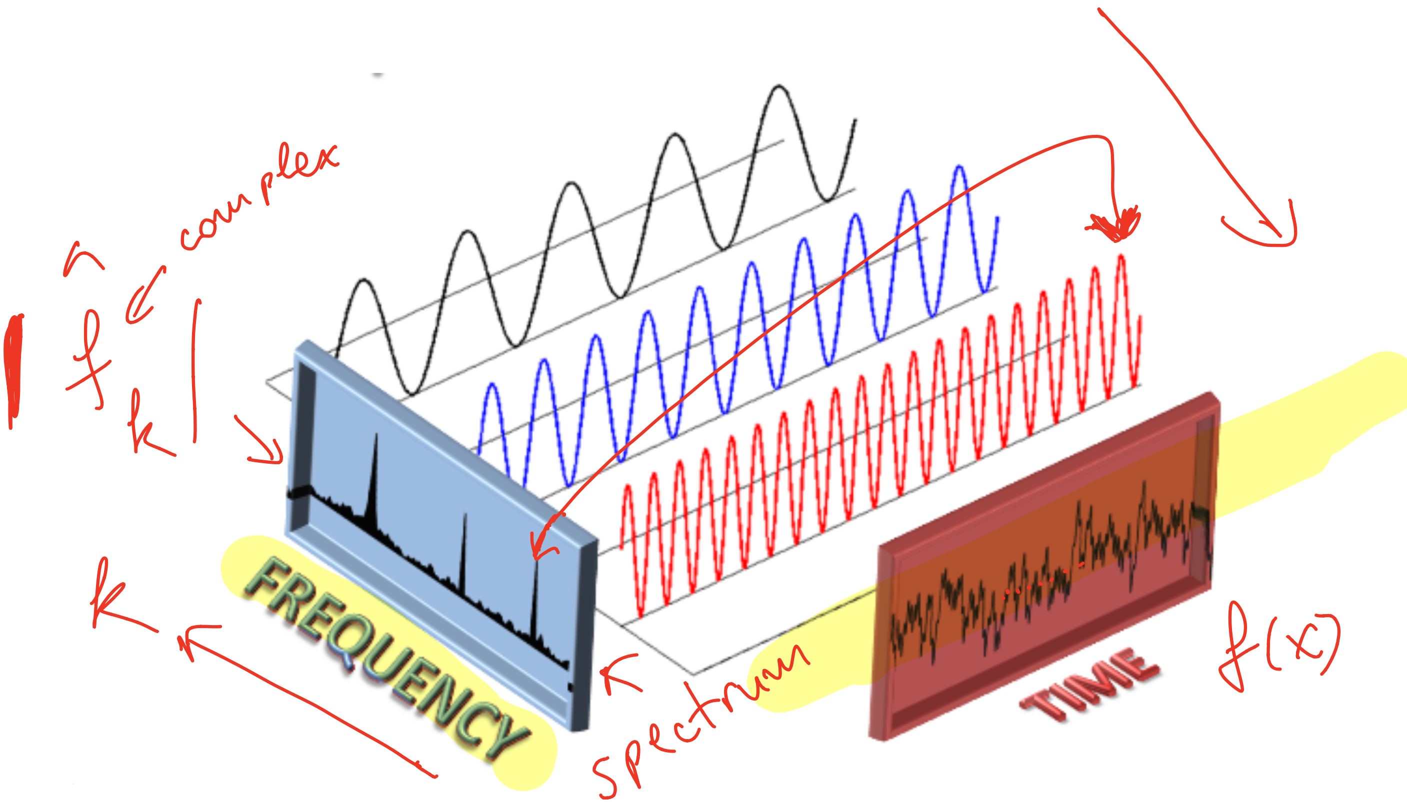
$$\forall f \in L^2_{2\pi}: \quad f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}, \Rightarrow \left\| f - \sum \hat{f}_k e^{ikx} \right\|_2 = 0$$

Fourier series

where the **Fourier coefficients** can be computed for any **frequency or wavenumber** k using:

$$\hat{f}_k = \frac{(f, \phi_k)}{2\pi} = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(x)e^{-ikx}dx.$$

Fourier Decomposition



Truncated Fourier Basis

- For a general interval $[0, X]$ the **discrete frequencies** are

$$k = \frac{2\pi}{X}\kappa \quad \kappa = 0, \pm 1, \pm 2, \ldots$$

- For non-periodic functions one can take the limit $X \to \infty$ in which case we get **continuous frequencies**.
- Now consider a **discrete Fourier basis** that only includes the first N basis functions, i.e.,

$$\begin{cases} k = -(N-1)/2, \ldots, 0, \ldots, (N-1)/2 & \text{if } N \text{ is odd} \\ k = -N/2, \ldots, 0, \ldots, N/2 - 1 & \text{if } N \text{ is even,} \end{cases}$$

and for simplicity we focus on N odd.

- The least-squares **spectral approximation** for this basis is:

$$f(x) \approx \phi(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k e^{ikx}.$$

Discrete Fourier Basis

- Let us discretize a given function on a set of N **equi-spaced nodes** as a vector

$$\mathbf{f}_j = f(x_j) \quad \text{where} \quad x_j = jh \text{ and } h = \frac{2\pi}{N}.$$

  Observe that $j = N$ **is the same node as** $j = 0$ **due to periodicity** so we only consider N instead of $N+1$ nodes.

- Now consider a **discrete Fourier basis** that only includes the first N basis functions, i.e.,

$$\begin{cases} k = -(N-1)/2, \ldots, 0, \ldots, (N-1)/2 & \text{if } N \text{ is odd} \\ k = -N/2, \ldots, 0, \ldots, N/2 - 1 & \text{if } N \text{ is even.} \end{cases}$$

- Focus on N odd and denote $K = (N-1)/2$.
- **Discrete dot product** between discretized "functions":

$$\mathbf{f} \cdot \mathbf{g} = h \sum_{j=0}^{N-1} f_j g_j^\star$$

Fourier Interpolant

$$\forall f \in L^2_{2\pi}: \quad f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$$

- We will try to approximate periodic functions with a **truncated Fourier series**:

$$f(x) \approx \phi(x) = \sum_{k=-K}^{K} \phi_k(x) = \sum_{k=-K}^{K} \hat{f}_k e^{ikx}.$$

- The discrete Fourier basis is $\{\phi_{-K}, \ldots, \phi_K\}$,

$$(\phi_k)_j = \exp(ikx_j),$$

and it is a **discretely orthonormal basis** in which we can represent periodic functions,

$$\phi_k \cdot \phi_{k'} = 2\pi\delta_{k,k'}$$

$e^{ikx_j} \; e^{-ikx_j}$

Proof of Discrete Orthogonality

The case $k = k'$ is trivial, so focus on

$$\phi_k \cdot \phi_{k'} = 0 \text{ for } k \neq k'$$

$$\sum_j \exp(ikx_j)\exp(-ik'x_j) = \sum_j \exp[i(\Delta k)x_j] = \sum_{j=0}^{N-1} [\exp(ih(\Delta k))]^j$$

where $\Delta k = k - k'$. This is a geometric series sum:

$$\phi_k \cdot \phi_{k'} = \frac{1 - z^N}{1 - z} = 0 \text{ if } k \neq k'$$

since $z = \exp(ih(\Delta k)) \neq 1$ and
$z^N = \exp(ihN(\Delta k)) = \exp(2\pi i(\Delta k)) = 1$.

Fourier Matrix

- Let us collect the discrete Fourier basis functions as columns in a **unitary** $N \times N$ **matrix** (*fft*(*eye*(N)) in MATLAB)

$$\boldsymbol{\Phi}_N = \left[\phi_{-K} \middle| \dots \phi_0 \dots \middle| \phi_K \right] \quad \Rightarrow \quad \Phi_{jk}^{(N)} = \frac{1}{\sqrt{N}} \exp\left(2\pi ijk/N\right)$$

- The truncated Fourier series is

$$\mathbf{f} = \boldsymbol{\Phi}_N \hat{\mathbf{f}}.$$

$\hat{f} \to f$

- Since the matrix $\boldsymbol{\Phi}_N$ is unitary, we know that $\boldsymbol{\Phi}_N^{-1} = \boldsymbol{\Phi}_N^{\star}$ and therefore

$$\hat{\mathbf{f}} = \boldsymbol{\Phi}_N^{\star} \mathbf{f},$$

$f \to \hat{f}$

which is nothing more than a change of basis!

Discrete Fourier Transform

- The **Fourier interpolating polynomial** is thus easy to construct

$$\phi_N(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k^{(N)} e^{ikx}$$

where the **discrete Fourier coefficients** are given by

$$\hat{f}_k^{(N)} = \frac{\mathbf{f} \cdot \phi_k}{2\pi} = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \exp(-ikx_j) \approx \hat{f}_k$$

- We can make the expressions more symmetric if we shift the frequencies to $k = 0, \ldots, N$, but one should still think of half of the frequencies as "negative" and half as "positive". See MATLAB's functions *fftshift* and *ifftshift*.

Discrete Fourier Transform

- The **Discrete Fourier Transform** (DFT) is a change of basis taking us from real/time to Fourier/frequency domain:

$$\text{Forward } \mathbf{f} \rightarrow \hat{\mathbf{f}} : \quad \hat{f}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right), \quad k = 0, \ldots, N-1$$

1 in matlab

$$\text{Inverse } \hat{\mathbf{f}} \rightarrow \mathbf{f} : \quad f_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{f}_k \exp\left(\frac{2\pi ijk}{N}\right), \quad j = 0, \ldots, N-1$$

N in matlab

- There is **different conventions** for the DFT depending on the interval on which the function is defined and placement of factors of N and 2π.
  Read the documentation to be consistent!
- A **direct** matrix-vector multiplication algorithm therefore takes $O(N^2)$ multiplications and additions. **Can we do it faster?** FFT yes!

Discrete spectrum

- The set of discrete Fourier coefficients $\hat{\mathbf{f}}$ is called the **discrete spectrum**, and in particular,

$$S_k = \left|\hat{f}_k\right|^2 = \hat{f}_k \hat{f}_k^\star,$$

  is the **power spectrum** which measures the frequency content of a signal.
- If f is real, then $\hat{f}$ satisfies the **conjugacy property**

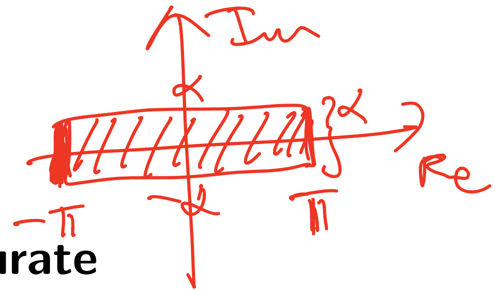
$$\hat{f}_{-k} = \hat{f}_k^\star,$$

  so that half of the spectrum is redundant and $\hat{f}_0$ is real.
- For an even number of points N the largest frequency $k = -N/2$ does not have a conjugate partner.

Approximation error: Analytic

- If $f\,(t = x + iy)$ is **analytic** in a half-strip around the real axis of half-width α and bounded by $|f\,(t)| < M$, then

$$\left|\hat{f}_k\right| \leq Me^{-\alpha|k|}.$$



- Then the Fourier interpolant is **spectrally-accurate**

$$\|f - \phi\|_\infty \leq 4 \sum_{k=n+1}^{\infty} Me^{-\alpha k} = \frac{2Me^{-\alpha n}}{e^{\alpha} - 1} \text{ (geometric series sum)}$$

- The Fourier interpolating trigonometric polynomial is spectrally accurate and a really great approximation for (very) smooth functions.

Spectral Accuracy (or not)

- The Fourier interpolating polynomial $\phi(x)$ has **spectral accuracy**, i.e., exponential in the number of nodes N

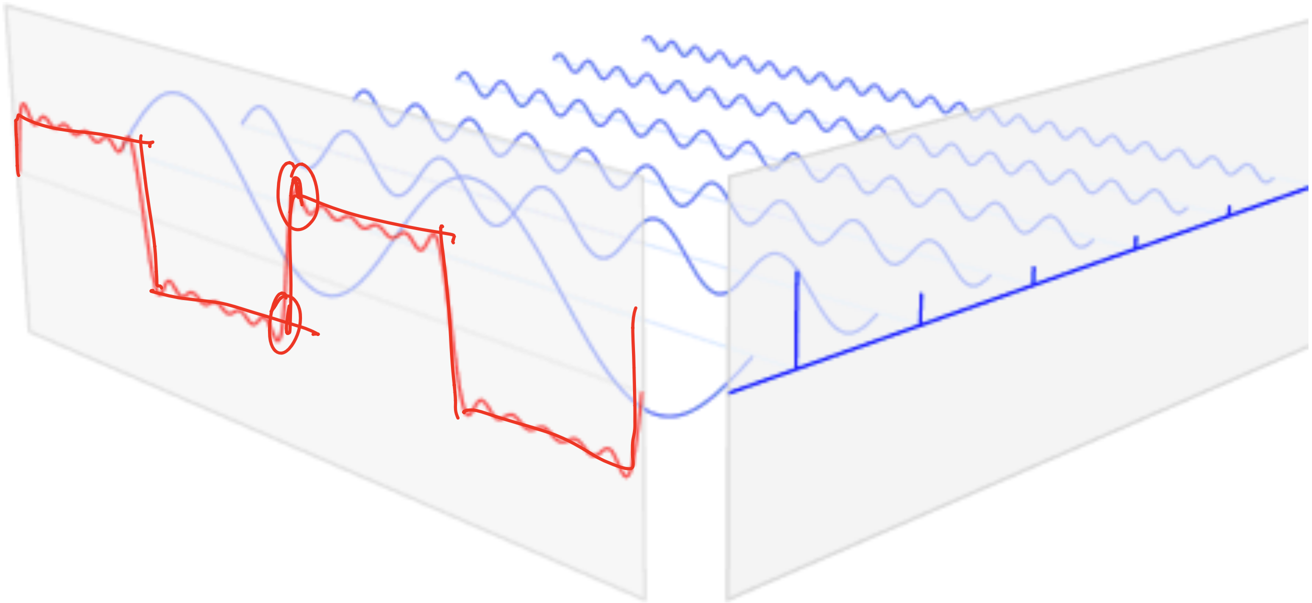
$$\|f(x) - \phi(x)\| \sim e^{-N}$$

for **sufficiently smooth functions**.

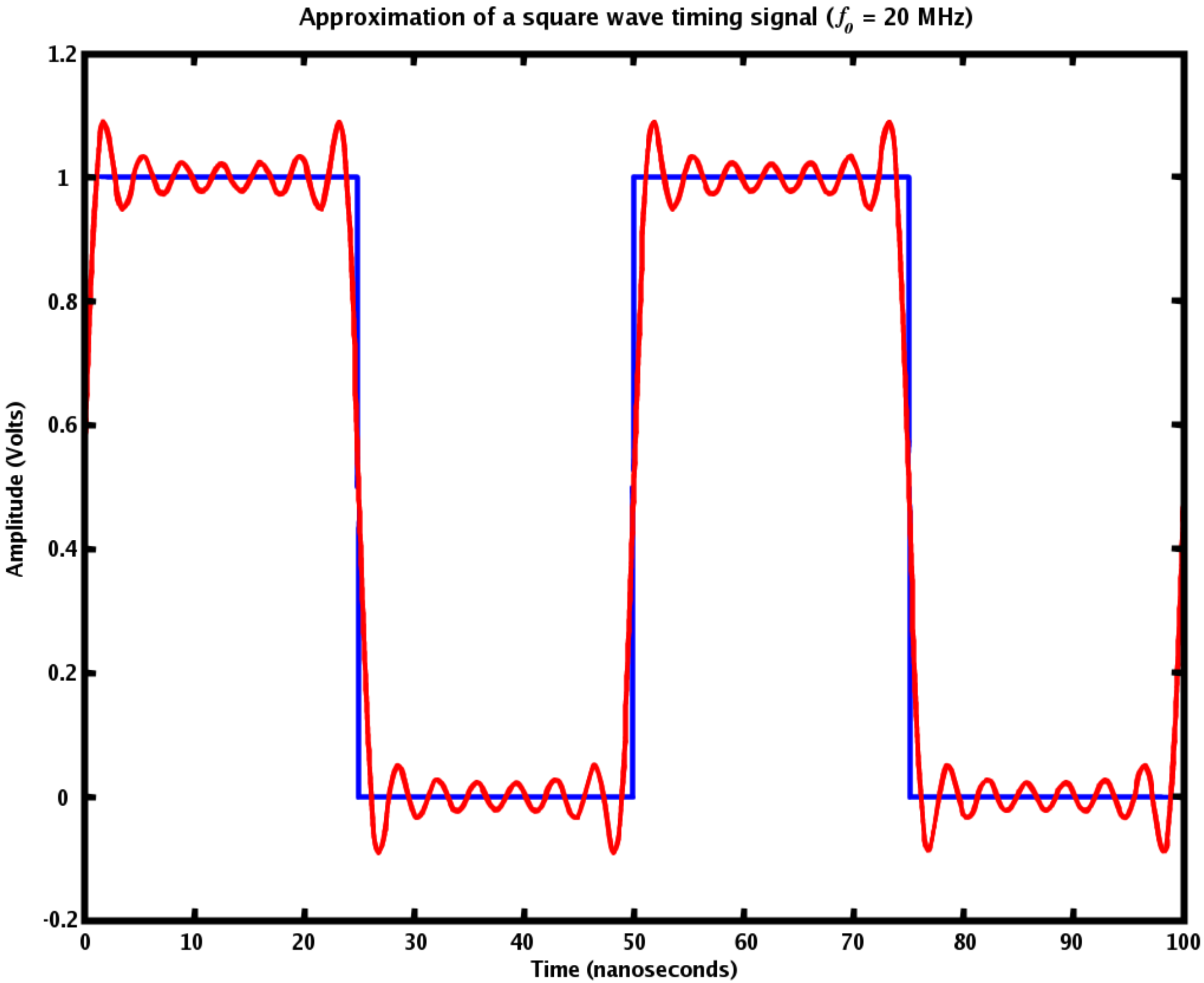
- Specifically, what is needed is sufficiently **rapid decay of the Fourier coefficients** with k, e.g., exponential decay $\left|\hat{f}_k\right| \sim e^{-|k|}$.
- Discontinuities cause slowly-decaying Fourier coefficients, e.g., power law decay $\left|\hat{f}_k\right| \sim k^{-1}$ for **jump discontinuities**.
- Jump discontinuities lead to slow convergence of the Fourier series for non-singular points (and no convergence at all near the singularity), so-called **Gibbs phenomenon** (ringing):

$$\|f(x) - \phi(x)\| \sim \begin{cases} N^{-1} & \text{at points away from jumps} \\ \text{const.} & \text{at the jumps themselves} \end{cases}$$

Gibbs Phenomenon

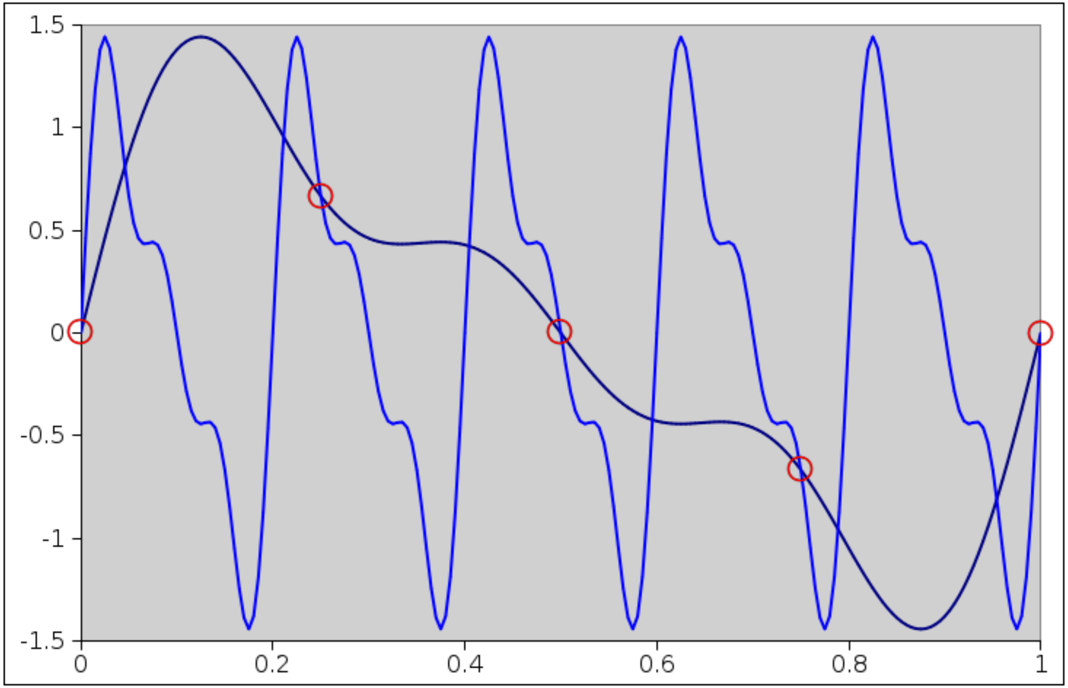


Gibbs Phenomenon



Aliasing

If we sample a signal at too few points the Fourier interpolant may be wildly wrong: **aliasing** of frequencies k and $2k, 3k, \ldots$



Standard anti-aliasing rule is the **Nyquist–Shannon** criterion (theorem): Need **at least 2 samples per period**.

DFT

- Recall the transformation from real space to frequency space and back:

$$\mathbf{f} \rightarrow \hat{\mathbf{f}}: \quad \hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right), \quad k = -\frac{(N-1)}{2}, \ldots, \frac{(N-1)}{2}$$

$$\hat{\mathbf{f}} \rightarrow \mathbf{f}: \quad f_j = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k \exp\left(\frac{2\pi ijk}{N}\right), \quad j = 0, \ldots, N-1$$

- We can make the forward-reverse **Discrete Fourier Transform** (DFT) more symmetric if we shift the frequencies to $k = 0, \ldots, N$:

Forward $\mathbf{f} \rightarrow \hat{\mathbf{f}}$:
$$\hat{f}_k = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right), \quad k = 0, \ldots, N-1$$

Inverse $\hat{\mathbf{f}} \rightarrow \mathbf{f}$:
$$f_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \hat{f}_k \exp\left(\frac{2\pi ijk}{N}\right), \quad j = 0, \ldots, N-1$$

FFT

- We can write the transforms in matrix notation:

$$\hat{\mathbf{f}} = \frac{1}{\sqrt{N}} \mathbf{U}_N \mathbf{f}$$

$$\mathbf{f} = \frac{1}{\sqrt{N}} \mathbf{U}_N^\star \hat{\mathbf{f}},$$

$\Phi = U$

where the **unitary Fourier matrix** is an $N \times N$ matrix with entries

$$e^{-2\pi ijk/N} = u_{jk}^{(N)} = \omega_N^{jk}, \quad \omega_N = e^{-2\pi i/N}.$$

- A **direct** matrix-vector multiplication algorithm therefore takes $O(N^2)$ multiplications and additions.
- Is there a faster way to compute the **non-normalized**

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j \omega_N^{jk} \;?$$

$k = 0, \ldots, N-1$

Can we do it in $O(N)$ time?

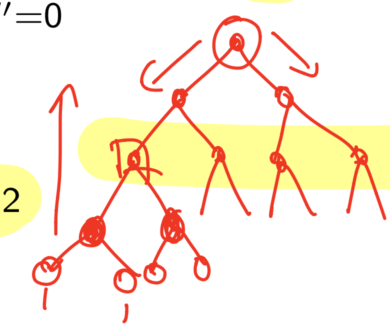
FFT

- For now assume that N is even and in fact a power of two, $N = 2^n$.
- The idea is to split the transform into two pieces, **even and odd** points:

$$\sum_{j=2j'} f_j \omega_N^{jk} + \sum_{j=2j'+1} f_j \omega_N^{jk} = \sum_{j'=0}^{N/2-1} f_{2j'} \left(\omega_N^2\right)^{j'k} + \omega_N^k \sum_{j'=0}^{N/2-1} f_{2j'+1} \left(\omega_N^2\right)^{j'k}$$

- Now notice that

$$\omega_N^2 = e^{-4\pi i/N} = e^{-2\pi i/(N/2)} = \omega_{N/2}$$



- This leads to a **divide-and-conquer algorithm**:

$$\hat{f}_k = \sum_{j'=0}^{N/2-1} f_{2j'} \omega_{N/2}^{j'k} + \omega_N^k \sum_{j'=0}^{N/2-1} f_{2j'+1} \omega_{N/2}^{j'k}$$

$$\hat{f}_k = \mathbf{U}_N \mathbf{f} = \left(\mathbf{U}_{N/2} \mathbf{f}_{even} + \omega_N^k \mathbf{U}_{N/2} \mathbf{f}_{odd}\right)$$

FFT Complexity

- The **Fast Fourier Transform** algorithm is **recursive**:

$$FFT_N(\mathbf{f}) = FFT_{\frac{N}{2}}(\mathbf{f}_{even}) + \mathbf{w} \boxdot FFT_{\frac{N}{2}}(\mathbf{f}_{odd}),$$

where $w_k = \omega_N^k$ and $\boxdot$ denotes element-wise product. When $N = 1$ the FFT is trivial (identity).

- To compute the whole transform we need $\log_2(N)$ steps, and at each step we only need N multiplications and $N/2$ additions at each step.
- The total **cost of FFT** is thus much better than the direct method's $O(N^2)$: **Log-linear**

$$O(N \log N).$$

- Even when N is not a power of two there are ways to do a similar **splitting** transformation of the large FFT into many smaller FFTs.
- Note that there are different **normalization conventions** used in different software.

In MATLAB

- The forward transform is performed by the function $\hat{f} = fft(f)$ and the inverse by $f = fft(\hat{f})$. Note that $ifft(fft(f)) = f$ and f and $\hat{f}$ may be complex.
- In MATLAB, and other software, the frequencies are not ordered in the "normal" way $-(N-1)/2$ to $+(N-1)/2$, but rather, the nonnegative frequencies come first, then the positive ones, so the "funny" ordering is

$$0, 1, \ldots, (N-1)/2, \quad -\frac{N-1}{2}, -\frac{N-1}{2}+1, \ldots, -1.$$

  This is because such ordering (shift) makes the forward and inverse transforms symmetric.
- The function *fftshift* can be used to order the frequencies in the "normal" way, and *ifftshift* does the reverse:

$$\hat{f} = fftshift(fft(f)) \text{ (normal ordering)}.$$

Multidimensional FFT

- DFTs and FFTs generalize straightforwardly to higher dimensions due to separability: **Transform each dimension independently**

$$\hat{f} = \frac{1}{N_x N_y} \sum_{j_y=0}^{N_y-1} \sum_{j_x=0}^{N_x-1} f_{j_x,j_y} \exp\left[-\frac{2\pi i \left(j_x k_x + j_y k_y\right)}{N}\right]$$

$$\hat{\mathbf{f}}_{k_x,k_y} = \frac{1}{N_x} \sum_{j_y=0}^{N_y-1} \exp\left(-\frac{2\pi i j_y k_x}{N}\right) \left[\frac{1}{N_y} \sum_{j_y=0}^{N_y-1} f_{j_x,j_y} \exp\left(-\frac{2\pi i j_y k_y}{N}\right)\right]$$

- For example, in two dimensions, **do FFTs of each column, then FFTs of each row of the result**:

$$\hat{\mathbf{f}} = \mathcal{F}_{row}\left(\mathcal{F}_{col}\left(\mathbf{f}\right)\right)$$

- The cost is N_y one-dimensional FFTs of length N_x and then N_x one-dimensional FFTs of length N_y:

$$N_x N_y \log N_x + N_x N_y \log N_y = N_x N_y \log\left(N_x N_y\right) = N \log N$$

Applications of FFTs

- Because FFT is a very fast, almost linear algorithm, it is used often to accomplish things that are not seemingly related to function approximation.
- Denote the Discrete Fourier transform, computed using FFTs in practice, with
$$\hat{\mathbf{f}} = \mathcal{F}(\mathbf{f}) \text{ and } \mathbf{f} = \mathcal{F}^{-1}\left(\hat{\mathbf{f}}\right).$$
- Plain FFT is used in signal processing for **digital filtering**: Multiply the spectrum by a filter $\hat{S}(k)$ discretized as $\hat{\mathbf{s}} = \left\{\hat{S}(k)\right\}_k$:
$$\mathbf{f}_{filt} = \mathcal{F}^{-1}\left(\hat{\mathbf{s}} \boxdot \hat{\mathbf{f}}\right).$$
- Examples include **low-pass**, **high-pass**, or **band-pass filters**. Note that **aliasing** can be a problem for digital filters.

FFT-based noise filtering (1)

```
Fs = 1000;                          % Sampling frequency
dt = 1/Fs;                          % Sampling interval
L = 1000;                           % Length of signal
t = (0:L-1)*dt;                     % Time vector
T=L*dt;                             % Total time interval

% Sum of a 50 Hz sinusoid and a 120 Hz sinusoid
x = 0.7*sin(2*pi*50*t) + sin(2*pi*120*t);
y = x + 2*randn(size(t));           % Sinusoids plus noise

figure(1); clf;
plot(t(1:100),y(1:100),'b--'); hold on
title('Signal Corrupted with Zero-Mean Random Noise')
xlabel('time')
```

FFT-based noise filtering (2)

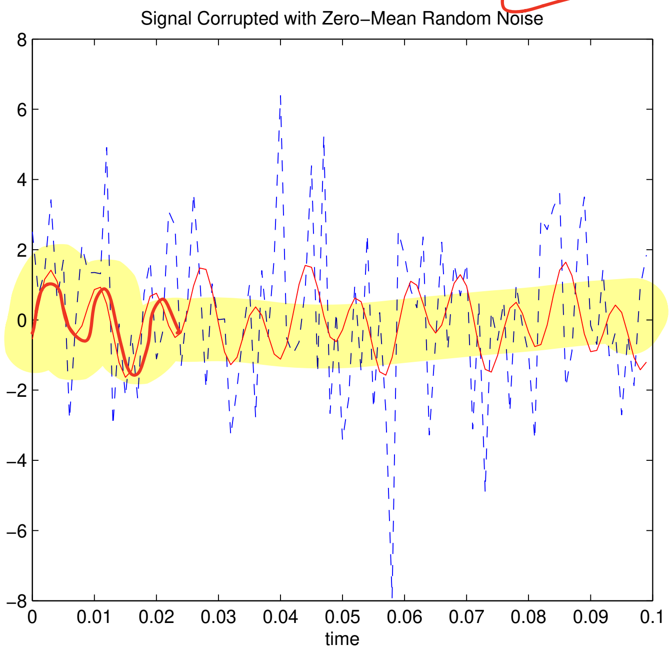
```
if (0)
   N=(L/2)*2; % Even N
   y_hat = fft(y(1:N));
   % Frequencies ordered in a funny way:
   f_funny = 2*pi/T* [0:N/2-1, -N/2:-1];
   % Normal ordering:
   f_normal = 2*pi/T* [-N/2 : N/2-1];
else
   N=(L/2)*2-1; % Odd N
   y_hat = fft(y(1:N));
   % Frequencies ordered in a funny way:
   f_funny = 2*pi/T* [0:(N-1)/2, -(N-1)/2:-1];
   % Normal ordering:
   f_normal = 2*pi/T* [-(N-1)/2 : (N-1)/2];
end
```

or fftshift(fft(y(1:N)))

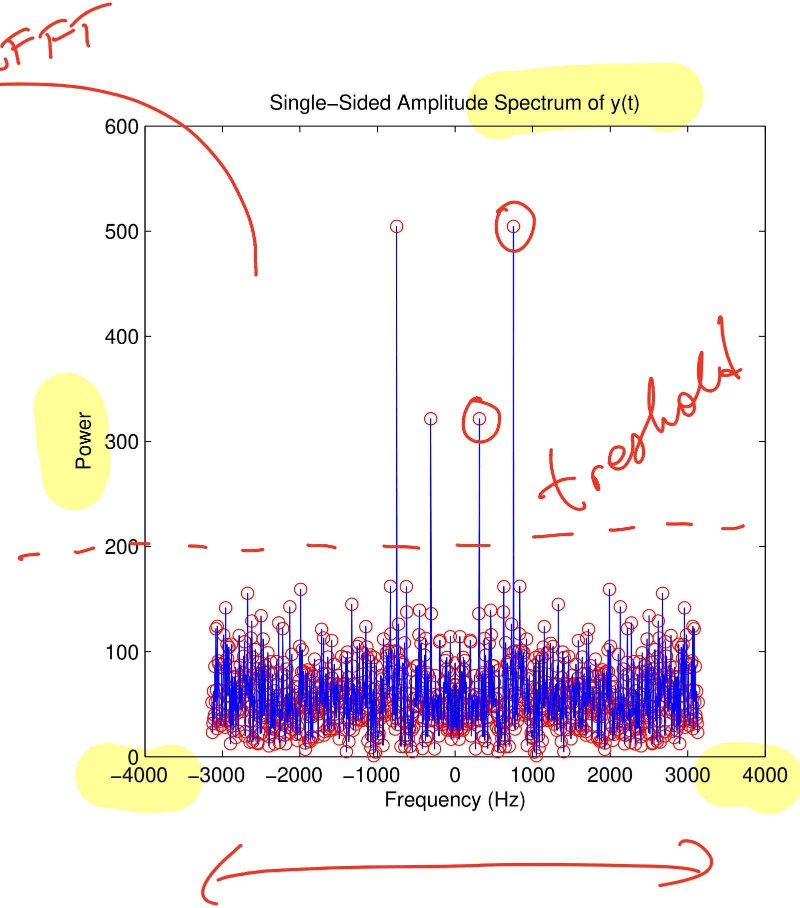
FFT-based noise filtering (3)

```
figure(2); clf; plot(f_funny, abs(y_hat), 'ro'); hold

y_hat=fftshift(y_hat);
figure(2); plot(f_normal, abs(y_hat), 'b-');

title('Single-Sided Amplitude Spectrum of y(t)')
xlabel('Frequency (Hz)')
ylabel('Power')

y_hat(abs(y_hat)<250)=0; % Filter out noise
y_filtered = ifft(ifftshift(y_hat));
figure(1); plot(t(1:100),y_filtered(1:100),'r-')
```

FFT results



Signal Corrupted with Zero−Mean Random Noise
8
6
4
2
0
−2
−4
−6
−8
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
time



iFFT
Single−Sided Amplitude Spectrum of y(t)
600
500
400
300
200
100
0
Power
treshold
−4000
−3000
−2000
−1000
0
1000
2000
3000
4000
Frequency (Hz)

Spectral Derivative

- Consider approximating the derivative of a periodic function $f(x)$, computed at a set of N equally-spaced nodes, $\mathbf{f}$.
- One way to do it is to use the **finite difference approximations**:

$$f'(x_j) \approx \frac{f(x_j + h) - f(x_j - h)}{2h} = \frac{f_{j+1} - f_{j-1}}{2h}.$$

- In order to achieve spectral accuracy of the derivative, we can differentiate the spectral approximation:
Spectrally-accurate finite-difference derivative

$$f'(x) \approx \phi'(x) = \frac{d}{dx}\phi(x) = \frac{d}{dx}\left(\sum_{k=0}^{N-1} \hat{f}_k e^{ikx}\right) = \sum_{k=0}^{N-1} \hat{f}_k \frac{d}{dx} e^{ikx}$$

$$\phi' = \sum_{k=0}^{N-1} \left(ik\hat{f}_k\right) e^{ikx} = \mathcal{F}^{-1}\left(i\hat{\mathbf{f}} \boxdot \mathbf{k}\right)$$

- **Differentiation becomes multiplication in Fourier space.**

Unmatched mode

- Recall that for even N there is one unmatched mode, the one with the highest frequency and amplitude $\hat{f}_{N/2}$.
- We need to choose what we want to do with that mode; see notes by S. G. Johnson (MIT) linked on webpage for details:

$$\phi(x) = \hat{f}_0 + \sum_{0<k<N/2} \left(\hat{f}_k e^{ikx} + \hat{f}_{N-k} e^{-ikx} \right) + \hat{f}_{N/2} \cos\left(\frac{Nx}{2}\right).$$

This is the unique "**minimal oscillation**" trigonometric interpolant.

- Differentiating this we get

$$\widehat{(\phi')}_k = \hat{f}_k \begin{cases} 0 & \text{if } k = N/2 \\ ik & \text{if } k < N/2 \\ i\,(k-N) & \text{if } k > N/2 \end{cases}.$$

- Real valued interpolation samples result in **real-valued** $\phi(x)$ for all x.

FFT-based differentiation

```
% From Nick Trefethen's Spectral Methods book
% Differentiation of exp(sin(x)) on (0,2*pi]:
  N = 8; % Even number!
  h = 2*pi/N; x = h*(1:N)';
  v = exp(sin(x)); vprime = cos(x).*v;
  v_hat = fft(v);
  ik = 1i*[0:N/2-1 0 -N/2+1:-1]'; % Zero special mode
  w_hat = ik .* v_hat;
  w = real(ifft(w_hat));
  error = norm(w-vprime,inf)
```

(no fftshift)

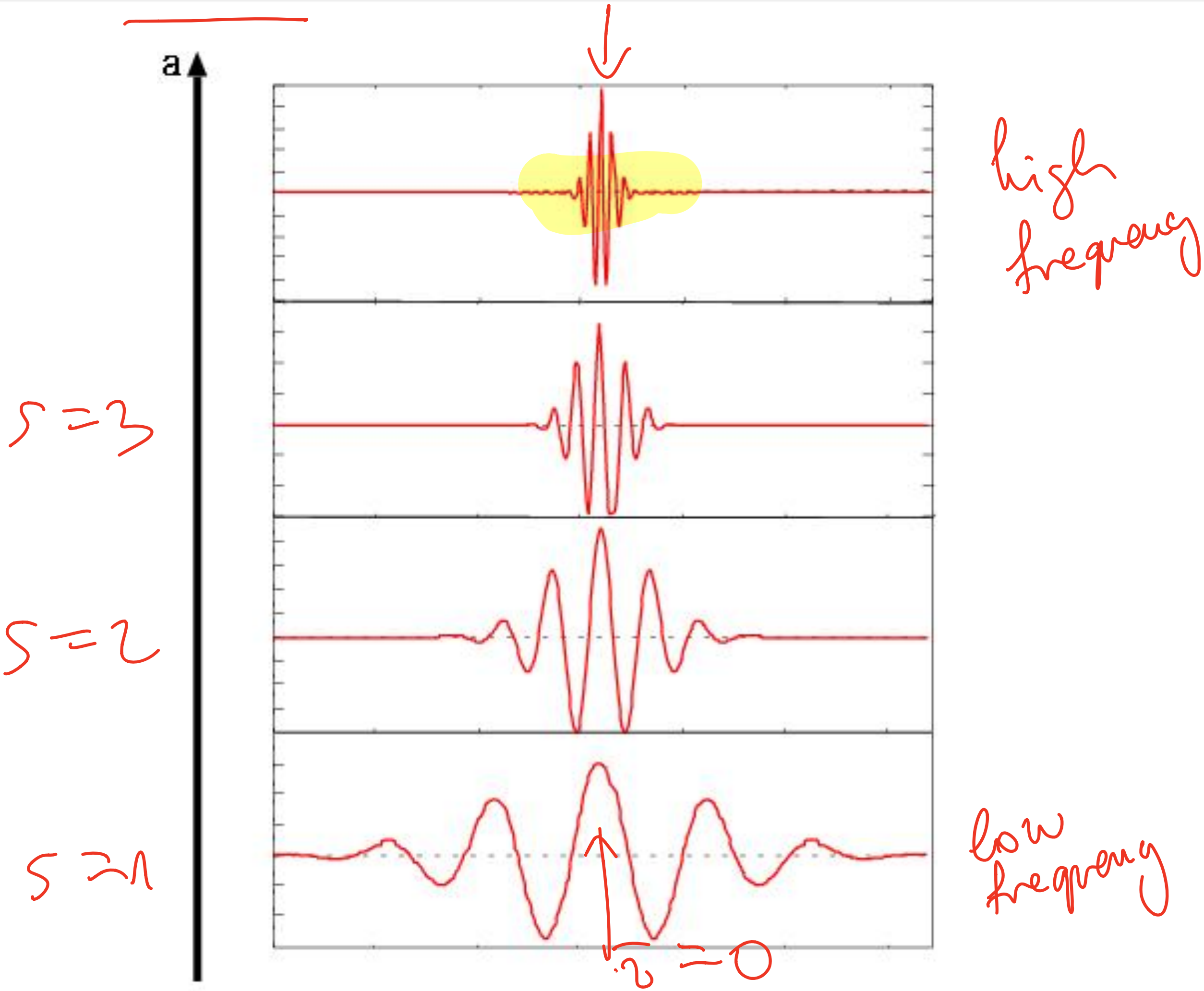
integral = ifft $\left(\frac{1}{ik} .* \text{fft}(f) \right)$

↑ except $k=0$

The need for wavelets

- Fourier basis is great for analyzing periodic signals, but is not good for functions that are **localized in space**, e.g., brief bursts of speach.
- Fourier transforms are not good with handling **discontinuities** in functions because of the Gibbs phenomenon.
- Fourier polynomails **assume periodicity** and are not as useful for non-periodic functions.
- Because Fourier basis is not localized, the highest frequency present in the signal must be used everywhere: One cannot use **different resolutions in different regions of space**.

An example wavelet



Wavelet basis

- A **mother wavelet function** $W(x)$ is a localized function in space. For simplicity assume that $W(x)$ has compact support on $[0, 1]$.
- A **wavelet basis** is a collection of **wavelets** $W_{s,\tau}(x)$ obtained from $W(x)$ by **dilation** with a **scaling factor** s and **shifting** by a **translation factor** τ:

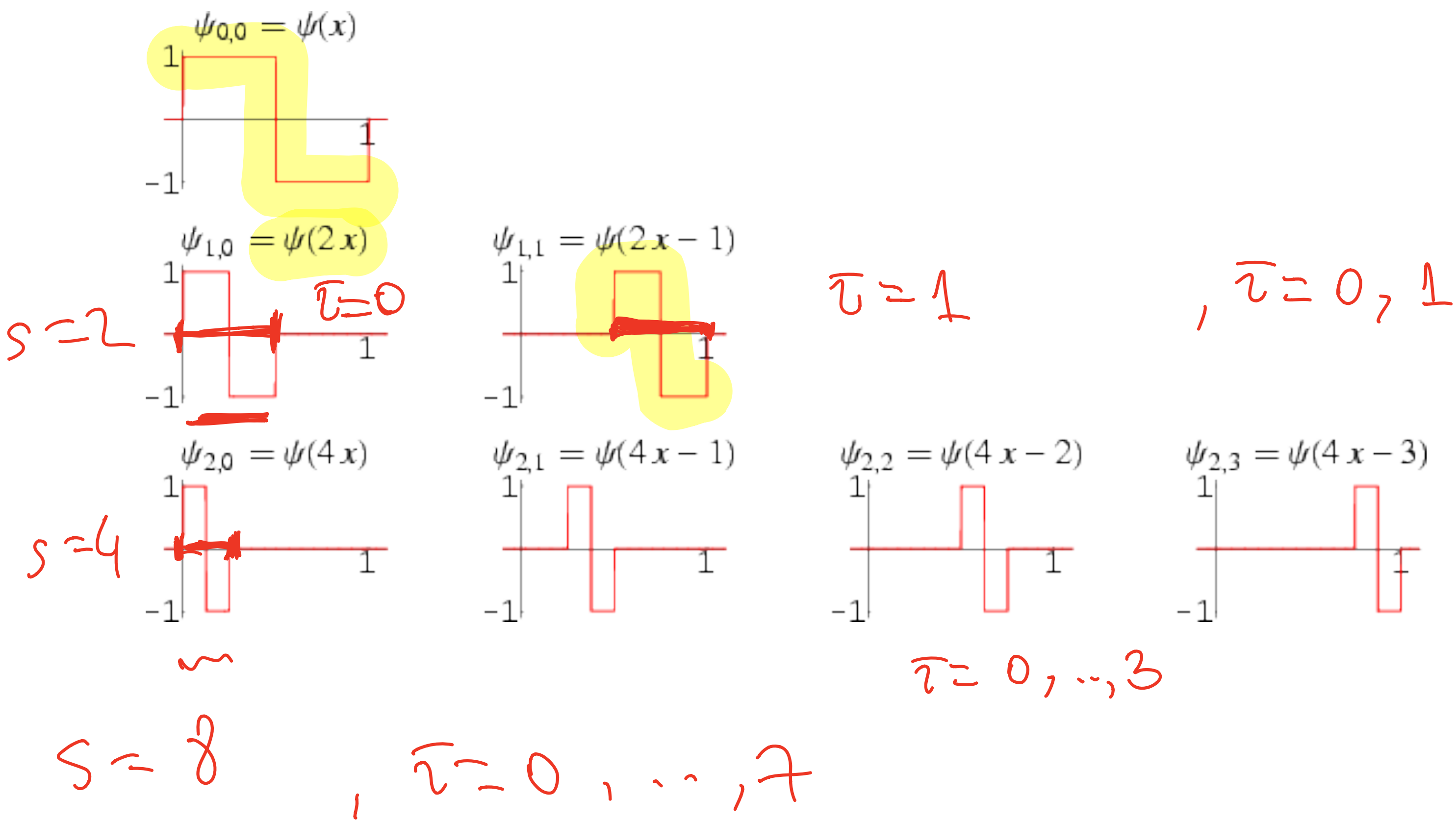
$$W_{s,\tau}(x) = W\left(sx - \tau\right).$$

frequency, location; e.g. Debauchies wavelet

- Here the scale plays the role of frequency in the FT, but the shift is novel and localized the basis functions in space.
- We focus on **discrete wavelet basis**, where the scaling factors are chosen to be powers of 2 and the shifts are integers:

$$W_{j,k} = W(2^j x - k), \quad k \in \mathbb{Z},\ j \in \mathbb{Z},\ j \geq 0.$$

Haar Wavelet Basis



Wavelet Transform

- Any function can now be represented in the wavelet basis:

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} W_{j,k}(x)$$

This representation picks out frequency components in different spatial regions.

- As usual, we truncate the basis at $j < J$, which leads to a total number of coefficients c_{jk}:

$$\sum_{j=0}^{J-1} 2^j = 2^J$$

Discrete Wavelet Basis

- Similarly, we discretize the function on a set of $N = 2^J$ equally-spaced nodes $x_{j,k}$ or intervals, to get the vector **f**:

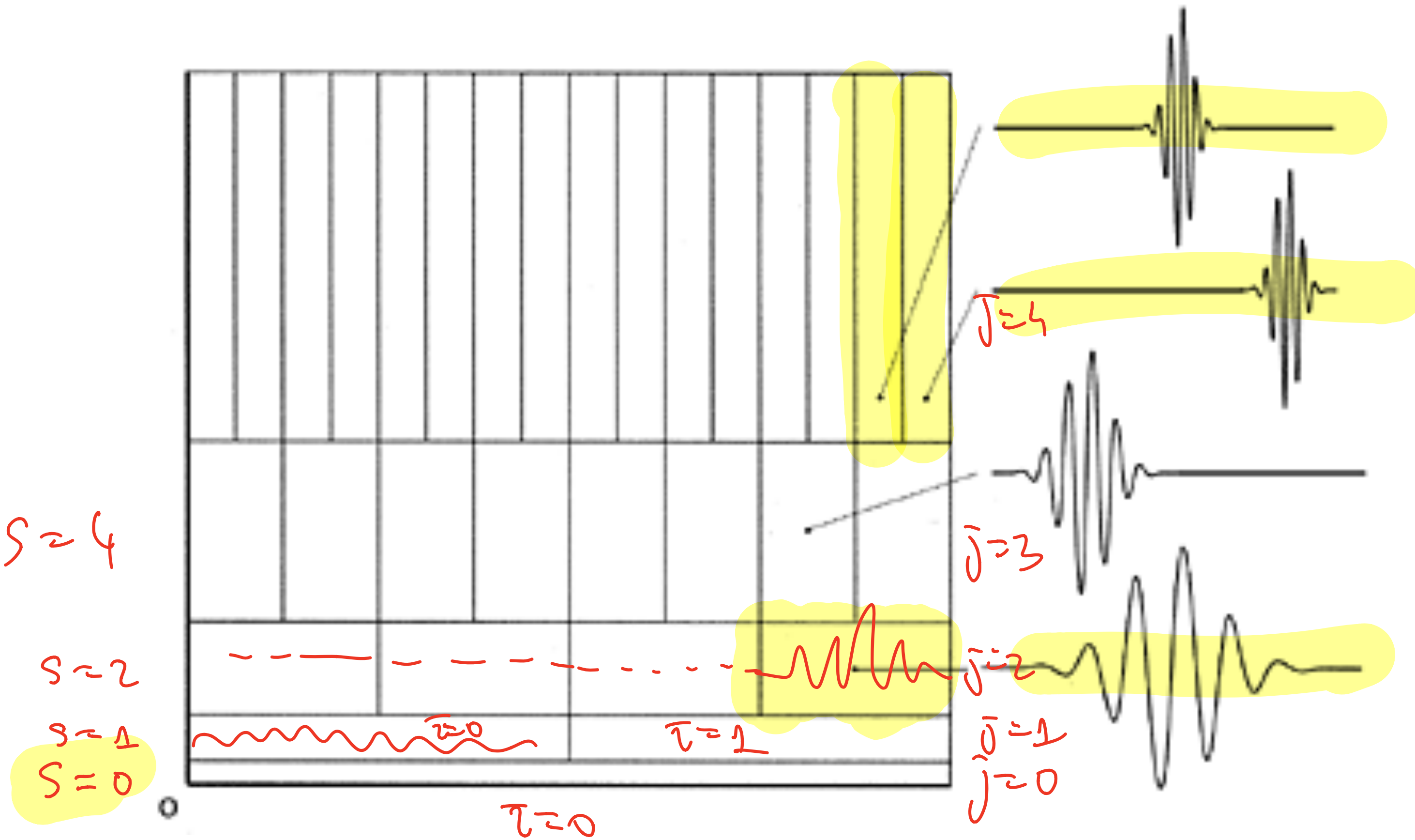
$$\mathbf{f} = c_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} c_{jk} W_{j,k}(x_{j,k}) = \mathbf{W}_j \mathbf{c}$$

- In order to be able to quickly and stably compute the coefficients **c** we need an **orthogonal wavelet basis**:

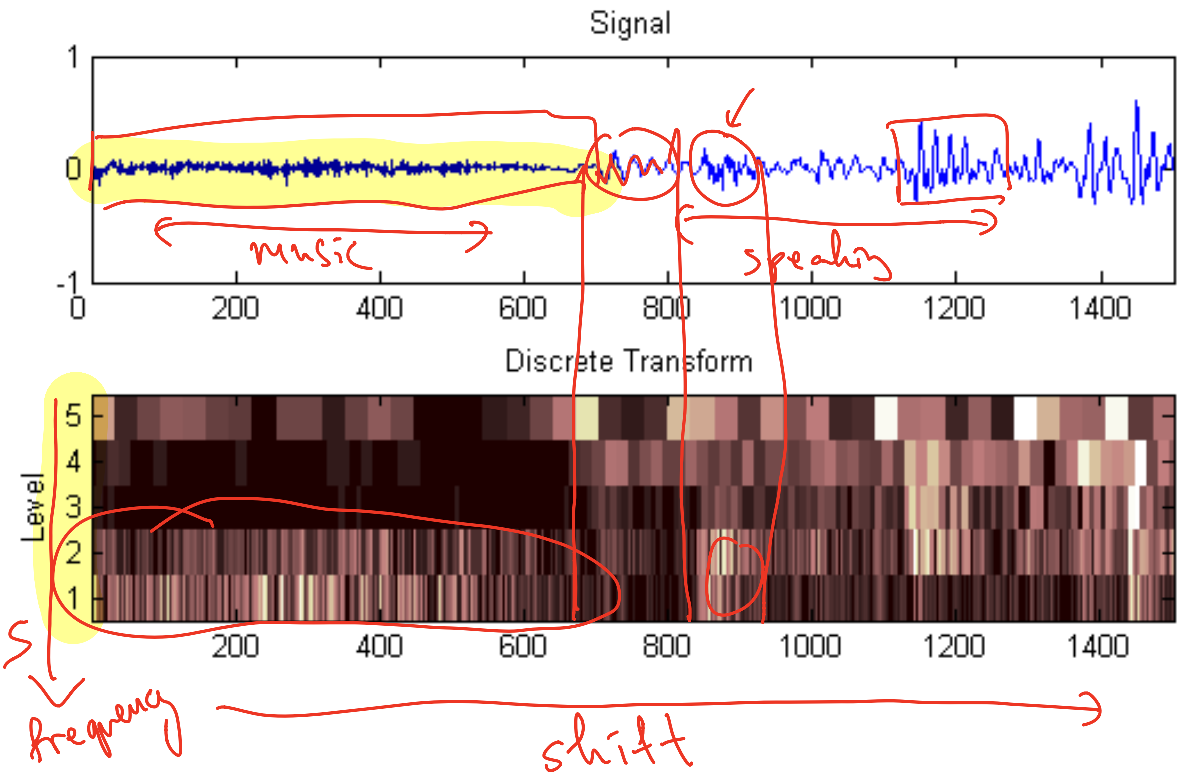
$$\int W_{j,k}(x) W_{l,m}(x) dx = \delta_{j,l} \delta_{l,m}$$

- The Haar basis is discretely orthogonal and computing the transform and its inverse can be done using a **fast wavelet transform**, in **linear time** $O(N)$ time.

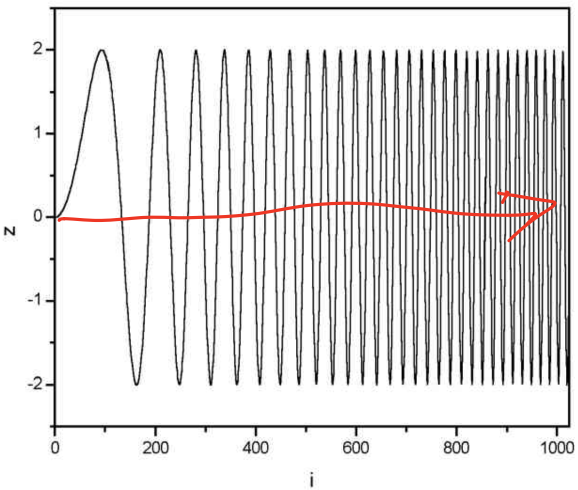
Discrete Wavelet Transform

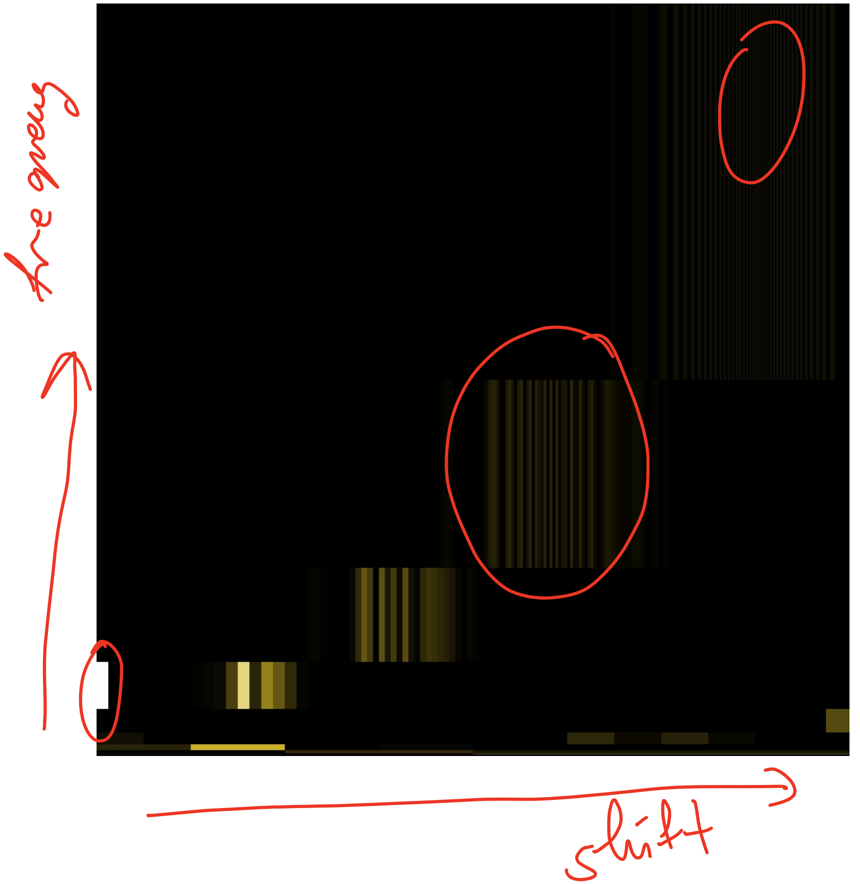


Scaleogram


Signal
1
0
-1
0
200
400
600
800
1000
1200
1400
music
speaking
Discrete Transform
Level
5
4
3
2
1
200
400
600
800
1000
1200
1400
frequency
shift

Another scaleogram





Conclusions/Summary

- **Periodic functions** can be approximated using basis of **orthogonal trigonometric polynomials**.
- The Fourier basis is **discretely orthogonal** and gives **spectral accuracy** for smooth functions.
- Functions with discontinuities are not approximated well: **Gibbs phenomenon**.
- The **Discrete Fourier Transform** can be computed very efficiently using the **Fast Fourier Transform** algorithm: $O(N \log N)$.
- FFTs can be used to **filter** signals, to do **convolutions**, and to provide spectrally-accurate **derivatives**, all in $O(N \log N)$ time.
- For signals that have different properties in different parts of the domain a **wavelet basis** may be more appropriate.
- Using specially-constructed **orthogonal discrete wavelet basis** one can compute **fast discrete wavelet transforms** in time $O(N)$.